

# Tau Decay Determination of the QCD Coupling

Antonio Pich

IFIC, Univ. València–CSIC, València, Spain & Physics Dep., TUM, Munich, Germany

The inclusive character of the total  $\tau$  hadronic width renders possible [1] an accurate calculation of the ratio  $R_\tau \equiv \Gamma[\tau^- \rightarrow \nu_\tau \text{ hadrons}]/\Gamma[\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e]$ . Its Cabibbo-allowed component can be written as [2]

$$R_{\tau,V+A} = N_C |V_{ud}|^2 S_{\text{EW}} \{1 + \delta_P + \delta_{\text{NP}}\}, \quad (1)$$

where  $N_C = 3$  is the number of quark colours and  $S_{\text{EW}} = 1.0201 \pm 0.0003$  contains the electroweak radiative corrections. The non-perturbative contributions are suppressed by six powers of the  $\tau$  mass [1] and can be extracted from the invariant-mass distribution of the final hadrons [3]. From the ALEPH data, one obtains  $\delta_{\text{NP}} = -0.0059 \pm 0.0014$  [4].

The dominant correction ( $\sim 20\%$ ) is the perturbative QCD contribution [1] [3]

$$\delta_P = \sum_{n=1} K_n A^{(n)}(\alpha_s) = \sum_{n=1} (K_n + g_n) a_\tau^n \equiv \sum_{n=1} r_n a_\tau^n, \quad (2)$$

which is determined by the coefficients of the perturbative expansion of the ( $N_F = 3$ ) QCD Adler function, already known to  $O(\alpha_s^4)$  [5]:  $K_0 = K_1 = 1$ ;  $K_2 = 1.63982$ ;  $K_3(\overline{MS}) = 6.37101$  and  $K_4(\overline{MS}) = 49.07570$ . The functions [3]

$$A^{(n)}(\alpha_s) = \frac{1}{2\pi i} \oint_{|s|=m_\tau^2} \frac{ds}{s} \left( \frac{\alpha_s(-s)}{\pi} \right)^n \left( 1 - 2\frac{s}{m_\tau^2} + 2\frac{s^3}{m_\tau^6} - \frac{s^4}{m_\tau^8} \right) = a_\tau^n + \mathcal{O}(a_\tau^{n+1}) \quad (3)$$

are contour integrals in the complex plane, which only depend on  $a_\tau \equiv \alpha_s(m_\tau^2)/\pi$ . Using the exact solution (up to unknown  $\beta_{n>4}$  contributions) for  $\alpha_s(-s)$  given by the renormalization-group  $\beta$ -function equation, they can be numerically computed with very high accuracy [3].

If the integrals  $A^{(n)}(\alpha_s)$  are expanded in powers of  $a_\tau$ , one recovers the naive perturbative expansion of  $\delta_P$  shown in the rhs of Eq. (2). This approximation is known as *fixed-order perturbation theory* (FOPT), while the improved expression, keeping the non-expanded values of  $A^{(n)}(\alpha_s)$ , is usually called *contour-improved perturbation theory* (CIPT) [3]. Even at  $\mathcal{O}(a_\tau^4)$ , FOPT gives a rather bad approximation to the integrals  $A^{(n)}(\alpha_s)$ , overestimating  $\delta_P$  by 12% at  $a_\tau = 0.11$ . The long running of  $\alpha_s(-s)$  along the circle  $|s| = m_\tau^2$  generates very large  $g_n$  coefficients, which depend on  $K_{m<n}$  and  $\beta_{m<n}$  [3]:  $g_1 = 0$ ,  $g_2 = 3.56$ ,  $g_3 = 19.99$ ,  $g_4 = 78.00$ ,  $g_5 = 307.78$ . These corrections are much larger than the original  $K_n$  contributions, giving rise to a badly behaved perturbative series (at the four-loop level the expansion of  $\alpha_s(-s)$  in powers of  $a_\tau$  is only convergent for  $a_\tau < 0.11$ , which is very close to the physical value of  $a_\tau$ ). Thus, it seems compulsory to resum the large logarithms,  $\log^n(-s/m_\tau^2)$ , using the renormalization group. This is precisely what CIPT does.

It has been argued that in the asymptotic regime (large  $n$ ) the renormalon behaviour of the  $K_n$  coefficients could induce cancelations with the running  $g_n$  corrections, which would be missed by CIPT. In that case, FOPT could approach faster the ‘true’ result provided by the Borel summation of the full renormalon series. This happens actually in the large- $\beta_1$

limit, which however does not approximate well the known perturbative series (for  $n \leq 4$  the true  $K_n$  coefficients add constructively with the  $g_n$  contributions). Models of higher-order corrections which assume a precocious asymptotic behaviour of the Adler function already at  $n = 3, 4$  [6] [7] seem to favour the FOPT result. The CIPT procedure is much more reliable in all other scenarios.

The present experimental value  $R_{\tau,V+A} = 3.4771 \pm 0.0084$  [8] implies  $\delta_P = 0.2030 \pm 0.0033$ . The two different treatments of the perturbative series result in

$$\alpha_s(m_\tau^2)_{\text{CIPT}} = 0.3412 \pm 0.0041_{\delta_P} \begin{matrix} +0.0069 \\ -0.0064 \end{matrix}_{K_5} \begin{matrix} +0.0050 \\ -0.0001 \end{matrix}_{\mu} \begin{matrix} +0.0039 \\ -0.0034 \end{matrix}_{\beta_5} = 0.344 \pm 0.014, \quad (4)$$

$$\alpha_s(m_\tau^2)_{\text{FOPT}} = 0.3194 \pm 0.0028_{\delta_P} \begin{matrix} +0.0039 \\ -0.0035 \end{matrix}_{K_5} \begin{matrix} +0.0105 \\ -0.0045 \end{matrix}_{\mu} \begin{matrix} +0.0019 \\ -0.0045 \end{matrix}_{\beta_5} = 0.321 \pm 0.015. \quad (5)$$

Higher-order corrections have been estimated adding the fifth-order term  $K_5 A^{(5)}(\alpha_s)$  with  $K_5 = 275 \pm 400$ . We have also included the 5-loop variation with changes of the renormalization scale in the range  $\mu^2/(-s) \in [0.4, 2.0]$ . The error induced by the truncation of the  $\beta$  function at fourth order has been conservatively estimated through the variation of the results at five loops, assuming  $\beta_5 = \pm \beta_4^2/\beta_3 = \mp 443$ ; in CIPT this slightly changes the values of  $A^{(n)}(\alpha_s)$ , while in FOPT it increases the scale sensitivity. The FOPT result shows as expected [3] [9] a much more sizeable  $\mu$  dependence, but it gets smaller errors from  $\delta_P$  and  $K_5$ . The three theoretical uncertainties ( $K_5$ ,  $\mu$ ,  $\beta_5$ ) have been added linearly and their sum combined in quadrature with the ‘experimental’ error from  $\delta_P$ .

Combining the two results with the PDG prescription (scale factor  $S = 1.14$ ), one gets  $\alpha_s(m_\tau^2) = 0.334 \pm 0.011$ . We keep conservatively the smallest error, i.e.

$$\alpha_s(m_\tau^2) = 0.334 \pm 0.014 \quad \longrightarrow \quad \alpha_s(M_Z^2) = 0.1204 \pm 0.0016. \quad (6)$$

The resulting value is in excellent agreement with the direct measurement of  $\alpha_s(M_Z)$  at the  $Z$  peak, providing a very significant experimental verification of *asymptotic freedom*.

## References

- [1] E. Braaten, S. Narison and A. Pich, Nucl. Phys. B 373 (1992) 581. E. Braaten, Phys. Rev. Lett. 60 (1988) 1606; Phys. Rev. D 39 (1989) 1458. S. Narison and A. Pich, Phys. Lett. B 211 (1988) 183.
- [2] A. Pich, arXiv:1101.2107 [hep-ph], arXiv:1001.0389 [hep-ph].
- [3] F. Le Diberder and A. Pich, Phys. Lett. B 286 (1992) 147, B 289 (1992) 165.
- [4] M. Davier et al., Rev. Mod. Phys. 78 (2006) 1043; Eur. Phys. J. C 56 (2008) 305.
- [5] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Phys. Rev. Lett. 101 (2008) 012002.
- [6] M. Beneke and M. Jamin, JHEP 0809 (2008) 044.
- [7] I. Caprini and J. Fischer, Eur. Phys. J. C 64 (2009) 35; arXiv:1106.5336 [hep-ph].
- [8] Heavy Flavor Averaging Group (HFAG), <http://www.slac.stanford.edu/xorg/hfag/>.
- [9] S. Menke, arXiv:0904.1796 [hep-ph].